

# SEMICLASSICAL SINGLE AND DOUBLE LAYER POTENTIALS: BOUNDEDNESS AND SHARPNESS

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**ABSTRACT.** In this paper, we investigate semiclassical single and double layer potentials mapping boundary data to interior functions of a domain at high energy  $\lambda^2 \rightarrow \infty$ . On single layer potentials, we prove that they are bounded in general and curved domains of differential norms depending on  $\lambda$ . On double layer potentials, we prove that they are uniformly bounded. In the end, we show that the bounds in all the theorems are sharp.

## 1. INTRODUCTION

Denote  $\Delta = \sum_{i=1}^n \partial_i^2$  as the Laplacian operator in  $\mathbb{R}^n$ . Given a piecewise smooth and bounded domain  $\Omega \subset \mathbb{R}^n$  and an harmonic function  $u$ , Green's formula gives

$$u(x) = \int_{\partial\Omega} N(x-y) \partial_{\nu_y} u(y) d\sigma_y - \int_{\partial\Omega} \partial_{\nu_y} N(x-y) u(y) d\sigma_y.$$

Here,  $\partial_{\nu_y}$  is the outward normal derivative at  $y \in \partial\Omega$ ,  $d\sigma$  is the surface measure on  $\partial\Omega$ , and  $N$  is the fundamental solution for  $-\Delta$ , that is,  $-\Delta N(x) = \delta(x)$ . We define

$$\int_{\partial\Omega} N(x-y) f(y) d\sigma_y \quad \text{and} \quad \int_{\partial\Omega} \partial_{\nu_y} N(x-y) f(y) d\sigma_y$$

respectively as single and double layer potentials with moment  $f$ . They are two convolution-type operators with kernels  $N$  and  $\partial_{\nu} N$ , and every harmonic function  $u$  in  $\Omega$  can be represented using these layer potential operators acting on boundary data  $\partial_{\nu} u$  and  $u|_{\partial\Omega}$ . See [9, Chapter 3] for a complete treatment of classical layer potentials with respect to Laplacian.

The mapping properties of layer potentials from boundary data to interior functions and related boundary value problems have been studied extensively over the past century. See [7, 17, 27] and [9, Section 3.F] for a detailed discussion of the classical problem.

In this paper we focus our attention on semiclassical layer potentials: Green's formula yields that the solution to the Helmholtz equation in  $\Omega$

$$-\Delta u = \lambda^2 u$$

has the form

$$(1.1) \quad u(x) = \int_{\partial\Omega} K_{\lambda}(x-y) \partial_{\nu_y} u(y) d\sigma_y - \int_{\partial\Omega} \partial_{\nu_y} K_{\lambda}(x-y) u(y) d\sigma_y,$$

where  $K_{\lambda}$  is a fundamental solution to  $(-\Delta - \lambda^2)$ , that is,

$$(1.2) \quad (-\Delta - \lambda^2) K_{\lambda}(x) = \delta(x).$$

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In fact, we have

$$(1.3) \quad K_\lambda^+(x) = \frac{i}{4} \left( \frac{\lambda}{2\pi|x|} \right)^{\frac{n-2}{2}} H_{\frac{n-2}{2}}^{(1)}(\lambda|x|),$$

where  $H_{\frac{n-2}{2}}^{(1)}$  is the Hankel function of the first kind and order  $\frac{n-2}{2}$ .  $K_\lambda^+$  is the kernel of the outgoing resolvent

$$R_\lambda^+ = [-\Delta - (\lambda + i0)^2]^{-1}.$$

That is,

$$[R_\lambda^+(u)]^\wedge(\xi) = \frac{\hat{u}(\xi)}{|\xi|^2 - (\lambda + i0)^2},$$

in which

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(x) dx$$

is the Fourier transform of  $u$ .

It is preferable for much of this paper to work within the semiclassical setting with  $h = \lambda^{-1}$ . In this setting Laplacian eigenfunctions are solutions to

$$p(x, hD)u = (-h^2\Delta - 1)u = 0,$$

where  $p(x, hD)$  has symbol  $p(x, \xi) = |\xi|^2 - 1$ . Similarly (1.2) can be written as

$$p(x, hD)K_{h^{-1}}^+(x) = h^2\delta(x).$$

Hörmander's theory on propagation of singularities asserts that

$$(1.4) \quad WF_h(K) \setminus WF_h(g) \subset \{(x, \xi) \in T^*\mathbb{R}^n : p(x, \xi) = 0\} = \{(x, \xi) \in T^*\mathbb{R}^n : |\xi| = 1\} = S^*\mathbb{R}^n,$$

which means that  $WF_h(K) \setminus WF_h(g)$  is in the bicharacteristic variety of  $p$ , and furthermore it is invariant under the Hamiltonian flow  $\Phi^t$  of  $p$ . Here,  $WF_h$  is the semiclassical wavefront set,  $T^*\mathbb{R}^n$  and  $S^*\mathbb{R}^n$  are the cotangent and cosphere bundles of  $\mathbb{R}^n$ , and  $\Phi^t$  of  $p$  is the geodesic flow in  $\mathbb{R}^n$ . See [15, 28] for a complete discussion of the theory.

Therefore we define the semiclassical single layer potential  $S_\lambda^+$  as

$$S_\lambda^+(f) = R_\lambda^+(fd\sigma) = K_\lambda^+ * (fd\sigma),$$

and the semiclassical double layer potential operator  $D_\lambda^+$  as

$$D_\lambda^+(f) = \partial_\nu K_\lambda^+ * (fd\sigma).$$

Now (1.1) can be written

$$(1.5) \quad u = S_\lambda^+(\partial_\nu u) - D_\lambda^+(u|_{\partial\Omega}),$$

allowing us to construct interior eigenfunction from boundary data. In particular note that Dirichlet eigenfunctions can be written as

$$u = S_\lambda^+(\partial_\nu u)$$

and Neumann eigenfunctions as

$$u = D_\lambda^+(u|_{\partial\Omega})$$

The representation of eigenfunctions in (1.5) has applications in both theoretical and numerical studies. To prove the Quantum Ergodicity (QE) of boundary values of eigenfunctions, Hassell and Zelditch [13] expressed boundary traces of Dirichlet, Neumann, and Robin eigenfunctions as eigenfunctions of some integral operators produced by semiclassical layer potentials.

In a similar vein Toth and Zelditch [24, 25] recently applied these potentials to prove Quantum Ergodic Restriction (QER) theorems on interior hypersurfaces.

If the boundary  $\partial\Omega$  is analytic, then there exist analytic continuations of  $S_\lambda^+$  and  $D_\lambda^+$  in the Grauert tube (a complex neighborhood of the real domain). Such complexification enables the study of zeros of eigenfunctions in the complex region (instead of in the real region), where it has simpler characterization. For detailed discussion on the nodal intersection estimates, see Toth and Zelditch [23], El-Hajj and Toth [6].

In star shaped domains Barnett and Hassell [2] develop a numerical techniques for constructing Dirichlet eigenfunctions by solving a related eigenfunction problem on the boundary. They then use (1.5) to reconstruct interior eigenfunctions. Their techniques allow them to control error on the boundary and so mapping norms on  $S_\lambda^+$  determine the control on error in the interior.

In [2, Remark 3.2], they proposed the question to find a bound on  $\lambda$ -dependence of the single layer potential  $S_\lambda^+$ . In particular they ask is

$$\|S_\lambda^+\|_{L^2(\partial\Omega) \rightarrow L^2(\Omega)} \lesssim \lambda^{-1}?$$

Such a bound would imply that boundary error controls interior error with no loss, which would be optimal for their numerical technique. In this paper we answer that question by obtaining bounds and sharp examples for the mapping properties of  $S_\lambda^+$ . We find that in general there is some loss on the optimal  $\lambda^{-1}$  bound however the extent of the loss depends on the geometry of  $\partial\Omega$ . Theorem 1.1 establishes the most general bounds which holds for all domains  $\Omega$ . Here we obtain a loss of  $\lambda^{1/4}$  over the optimal  $\lambda^{-1}$  bound.

**Theorem 1.1.**

$$\|S_\lambda^+(f)\|_{L^2(\Omega)} \leq c\lambda^{-\frac{3}{4}}\|f\|_{L^2(\partial\Omega)},$$

where  $c$  depends only on  $\Omega$ . Furthermore, the norm is sharp if the boundary  $\partial\Omega$  contains a flat piece.

**Remark.** We actually prove that the previous theorem holds for any compact set  $U \subset \mathbb{R}^n$  that

$$\|S_\lambda^+(f)\|_{L^2(U)} \leq c\lambda^{-\frac{3}{4}}\|f\|_{L^2(\partial\Omega)},$$

and it is sharp if the boundary  $\partial\Omega$  contains a flat piece.

One can similarly define the incoming resolvent outgoing resolvent

$$R_\lambda^- = [-\Delta - (\lambda - i0)^2]^{-1}$$

and  $K_\lambda^-$  as its kernel. Then the above theorem is apparently also valid with the same norm for  $S_\lambda^-$ . And we will drop the  $\pm$  sign in the  $L^2$  estimates without causing any confusion. Furthermore, if we write  $dE_\lambda$  as the spectral projection operator  $\delta(-\Delta - \lambda^2)$ , then Stone's formula (See e.g. [14, Chapter XIV].)

$$(1.6) \quad dE_\lambda = \frac{R_\lambda^+ - R_\lambda^-}{2\pi i}$$

immediately implies that

**Corollary 1.2.**

$$\|dE_\lambda(f d\sigma)\|_{L^2(\Omega)} \leq c\lambda^{-\frac{3}{4}}\|f\|_{L^2(\partial\Omega)}.$$

Furthermore, the norm is sharp if the boundary  $\partial\Omega$  contains a flat piece.

**Remark.**

1. Another description of  $R_\lambda^\pm$  and  $dE_\lambda$  is from the intersecting Lagrangian distribution theory introduced in [18]:  $R_\lambda^+$  is an intersecting Lagrangian distribution associated to two Lagrangian submanifolds:

- The conormal bundle to the diagonal,

$$L_1 = \{(x, \xi, y, \eta) \in T^*\mathbb{R}^n \times T^*\mathbb{R}^n : x = y, \xi = \eta\},$$

which is the lift of  $WF_h(g)$  in (1.4) from in  $T^*\mathbb{R}^n$  to in  $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ ;

- The bicharacteristic flowout  $\Phi^t$  in the positive direction from the intersection of  $L_1$  and the bicharacteristic variety  $S^*\mathbb{R}^n \times S^*\mathbb{R}^n$ ,

$$L_2 = \{(x, \xi, y, \eta) \in T^*\mathbb{R}^n \times T^*\mathbb{R}^n : \xi = \eta, |\eta| = 1, x = y + t\eta, t \geq 0\}.$$

$R_\lambda^-$  is the same except it would be the bicharacteristic flowout in the negative direction. When we subtract them in (1.6), the diagonal part cancels and  $dE_\lambda$  is associated to the flowout  $L_0$  in both directions, from the intersection of  $L_1$  and the characteristic variety:

$$L_0 = \{(x, \xi, y, \eta) \in T^*\mathbb{R}^n \times T^*\mathbb{R}^n : \xi = \eta, |\eta| = 1, x = y + t\eta, t \in \mathbb{R}\}.$$

2. The above characterization can also be seen in the proof of Theorem 1.1 in Section 2, as we cut the kernel into near-diagonal and off-diagonal parts, which correspond to the two Lagrangian submanifolds.

If the boundary of the domain  $\partial\Omega$  is curved, then Theorem 1.1 and Corollary 1.2 can be improved to the the following theorem.

**Theorem 1.3.** *If  $\partial\Omega$  is curved, that is, the second fundamental form of  $\partial\Omega$  is (positive or negative) definite, then*

$$\|S_\lambda(f)\|_{L^2(\Omega)} \leq c\lambda^{-\frac{5}{6}}\|f\|_{L^2(\partial\Omega)},$$

and

$$\|dE_\lambda(f d\sigma)\|_{L^2(\Omega)} \leq c\lambda^{-\frac{5}{6}}\|f\|_{L^2(\partial\Omega)},$$

where  $c$  depends only on  $\Omega$ . Furthermore, the norms are sharp if  $\Omega$  is an annulus.

**Remark.** The above two estimates are also valid for any compact  $U \subset \mathbb{R}^n$ :

$$\|S_\lambda(f)\|_{L^2(U)} \leq c\lambda^{-\frac{5}{6}}\|f\|_{L^2(\partial\Omega)},$$

and

$$\|dE_\lambda(f d\sigma)\|_{L^2(U)} \leq c\lambda^{-\frac{5}{6}}\|f\|_{L^2(\partial\Omega)}.$$

The improvement in curved domains can be understood by the propagation of semiclassical singularities and we provide a heuristic here:

The singularities of  $S_\lambda(f)$  and  $dE_\lambda(f d\sigma)$  propagate through the bicharacteristic flowout, i.e., the geodesic flow. Since  $f$  is supported on  $\partial\Omega$ , the dominating singularities propagate along the lines tangent to the boundary. Therefore it is natural to expect worse estimates in the flat case where tangent lines coincide with the boundary.

Turning our attention to double layer potentials, we prove

**Theorem 1.4.**

$$\|D_\lambda(f)\|_{L^2(\Omega)} \leq c\|f\|_{L^2(\partial\Omega)},$$

where  $c$  depends only on  $\Omega$ . Furthermore, the norm is sharp if  $\Omega$  is a disc.

**Remark.**

1. Weaker estimates in Theorems 1.1, 1.3, and 1.4 are obtained by Feng and Sheen [8] and Spence [19]. Precisely, Feng and Sheen [8, Theorem 3.4, Lemma 3.5, and Theorem 4.5] proved

$$\|S_\lambda(f)\|_{L^2(U)} \leq c\|f\|_{L^2(\partial\Omega)} \quad \text{and} \quad \|D_\lambda(f)\|_{L^2(U)} \leq \lambda\|f\|_{L^2(\partial\Omega)};$$

while Spence [19, Lemma 4.3] improved to

$$\|S_\lambda(f)\|_{L^2(U)} \leq c\lambda^{-\frac{1}{2}}\|f\|_{L^2(\partial\Omega)} \quad \text{and} \quad \|D_\lambda(f)\|_{L^2(U)} \leq \lambda^{\frac{1}{2}}\|f\|_{L^2(\partial\Omega)}.$$

Therefore, our results here improve further to the sharp cases. See also the survey article [4] for related results and their applications in numerical computations.

2. Contrary with the case of single layer potentials, we have uniform bound for double layer potentials in all domains. In this case tangential propagation of singularities is not an issue. This is because the kernel of  $D_\lambda^+$  is  $\partial_\nu K_\lambda^+$  and the semiclassical symbol of  $\lambda^{-1}\partial_\nu$  vanishes on the tangential directions.
3. In studying Quantum Ergodicity theory on the boundary, Hassell and Zelditch [13] used the double layer operator, that is, the mapping  $f \rightarrow D_\lambda(f)|_{\partial\Omega}$  from boundary data to boundary function. It would be interesting to estimate the norm of such mapping and understand its relation with Theorem 1.4, however we leave it for further investigation.

**1.1. Connection with boundary estimates of eigenfunctions.** Because of (1.5), there is a close relation between semiclassical layer potentials and boundary estimates of eigenfunctions.

- Dirichlet eigenfunction:  $u$  satisfies  $u = S_\lambda^+(\partial_\nu u)$ , and Hassell and Tao [11, 12] (see also Bardos, Lebeau, and Rauch [1]) proved that

$$\|u\|_{L^2(\Omega)} \approx \lambda^{-1} \|\partial_\nu u\|_{L^2(\partial\Omega)},$$

as  $u = S_\lambda^+(\partial_\nu u)$  this implies that

$$\|S_\lambda^+\|_{L^2(\partial\Omega) \rightarrow L^2(\Omega)} > c\lambda^{-1}.$$

Therefore the sharp examples for Theorem 1.1 that we produce in Section 4 are far from being normal derivatives of a Dirichlet eigenfunctions.

- Neumann eigenfunction:  $u$  satisfies  $u = D_\lambda^+(u|_{\partial\Omega})$ , and Tataru [22] proved that

$$(1.7) \quad \|u\|_{L^2(\partial\Omega)} \lesssim \lambda^{\frac{1}{3}} \|u\|_{L^2(\Omega)},$$

together with our Theorem 1.4, we conclude that

$$\|u\|_{L^2(\Omega)} \lesssim \|u\|_{L^2(\partial\Omega)} \lesssim \lambda^{\frac{1}{3}} \|u\|_{L^2(\Omega)}.$$

We will discuss the sharpness of above inequalities in Section 4, and construct examples for which the equalities are achieved.

**1.2. Connection with interior hypersurface restriction estimates of eigenfunctions.** Notice that the kernel of  $dE_\lambda$

$$\tilde{K}_\lambda = \frac{K_\lambda^+ - K_\lambda^-}{2\pi i}$$

satisfies  $(-\Delta - \lambda^2)\tilde{K}_\lambda = 0$ . Thus  $u = dE_\lambda(f d\sigma)$  is an eigenfunction in  $\mathbb{R}^n$ , and  $f = u|_{\partial\Omega}$  is the restriction of  $u$  on  $\partial\Omega$ . To estimate the norm of

$$dE_\lambda(\cdot d\sigma) : L^2(\partial\Omega) \rightarrow L^2(\Omega),$$

we consider the adjoint operator  $(dE_\lambda)^*$  with the same norm. In particular, the estimates of  $dE_\lambda$  in Corollary 1.2 and Theorem 1.3 are equivalent to

**Proposition 1.5.** *Given  $u \in L^2(\Omega)$ , we have*

$$\|(dE_\lambda)^*(u)\|_{L^2(\partial\Omega)} \leq c\lambda^{-\frac{3}{4}} \|u\|_{L^2(\Omega)}$$

*in general domain, and*

$$\|(dE_\lambda)^*(u)\|_{L^2(\partial\Omega)} \leq c\lambda^{-\frac{5}{6}} \|u\|_{L^2(\Omega)}$$

*if  $\partial\Omega$  is curved.*

*Proof of Proposition 1.5.* Write  $u_1 = (dE_\lambda)^*(u)$ , then  $u_1$  is an eigenfunction in  $\mathbb{R}^n$ . In particular, choose a compact set  $\Omega_1$  such that  $\Omega \Subset \Omega_1 \Subset \mathbb{R}^n$ , then  $u_1$  is an eigenfunction in  $\Omega_1$ , and  $\partial\Omega$  can be regarded as an interior hypersurface in  $\Omega_1$ . From the classical Fourier integral operator theory, (See [15, Chapter XXV].) we have

$$(1.8) \quad \|u_1\|_{L^2(\Omega_1)} \leq c\lambda^{-1}\|u\|_{L^2(\Omega)}.$$

Since  $\partial\Omega \subset \Omega$ , we can use the interior hypersurface restriction estimate of  $u_1$  from [3, 16]:

$$\|u_1\|_{L^2(\partial\Omega)} \leq c\lambda^{\frac{1}{4}}\|u_1\|_{L^2(\Omega)}$$

on general  $\partial\Omega$ , and

$$\|u_1\|_{L^2(\partial\Omega)} \leq c\lambda^{\frac{1}{6}}\|u_1\|_{L^2(\Omega)}$$

if  $\partial\Omega$  is curved.

Putting together with (1.8), we have the desired estimates in the proposition.  $\square$

**1.3. Connection with interior hypersurface estimates of quasimodes and strategy of the proofs.** Comparing with  $dE_\lambda(fd\sigma)$ ,  $S_\lambda^+(f)$  is an eigenfunction in  $\Omega$  and  $\mathbb{R}^n \setminus \Omega$ , but has a jump across the boundary  $\partial\Omega$ . To estimate the norm, we also consider the adjoint operator

$$(S_\lambda^+)^* : L^2(\Omega) \rightarrow L^2(\partial\Omega).$$

Since  $(S_\lambda^+)^*(u)$  for  $u \in L^2(\Omega)$  is no longer an eigenfunction in a larger domain, the interior hypersurface restriction estimates of eigenfunctions in Section 1.2 fail to apply to single layer potentials. However, after a suitable cutoff on the kernel of  $(S_\lambda^+)^*$ , it results an  $O_{L^2}(h)$  quasimode in  $\mathbb{R}^n$ , and thus subject to interior hypersurface restriction estimates of quasimodes in [20, 10]. We recall that an  $O_{L^2}(h)$  quasimode  $u(h)$  satisfies  $\|u(h)\|_{L^2(\mathbb{R}^n)} = O(1)$  and

$$\|p(x, hD)u(h)\|_{L^2(\mathbb{R}^n)} = O(h).$$

See [28, Section 7.4.1] for more details on quasimodes. So our strategy to prove the estimates on  $(S_\lambda^+)^*$  is to divide the kernel into near-diagonal and off-diagonal parts: The near-diagonal part admits better bound, and the off-diagonal part is reduced to estimating quasimodes.

A similar strategy applies to double layer potential  $(D_\lambda^+)^*$ , as the off-diagonal part on the boundary resembles the normal derivative of an  $O_{L^2}(h)$  quasimode, and therefore we can use the result on Neumann data restriction estimates in [21].

**Organisation of the paper.** In Section 2, we prove Theorems 1.1 and 1.3 and in Section 3 we prove Theorem 1.4. In Section 4, we show that all of these estimates are essentially sharp, and then give some further remarks concerning the relation between these bounds and the convexity of the domain.

Throughout this paper,  $A \lesssim B$  ( $A \gtrsim B$ ) means  $A \leq cB$  ( $A \geq cB$ ) for some constant  $c$  depending only on the domain;  $A \approx B$  means  $A \lesssim B$  and  $B \lesssim A$ ; the constants  $c$  and  $C$  may vary from line to line.

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## 2. BOUNDEDNESS OF SEMICLASSICAL SINGLE LAYER POTENTIALS

We aim to use previously know bounds for restriction of quasimode to hypersurfaces, both in the general and curved cases. We have that the single layer potential operator  $S_\lambda^+$  is given by

$$S_\lambda^+(f)(x) = \int_{\partial\Omega} K_\lambda(x-y)f(y)dy,$$

where

$$K_\lambda(x) = \frac{i}{4} \left( \frac{\lambda}{2\pi|x|} \right)^{\frac{n-2}{2}} H_{\frac{n-2}{2}}^{(1)}(\lambda|x|),$$

in which  $H_\beta^{(1)}$  is the Hankel function of the first kind and order  $\beta$ . Note that the adjoint of  $S_\lambda^+$  is given by

$$(2.1) \quad (S_\lambda^+)^* u(x) = \int_{\Omega} K_\lambda^*(x-y)u(y)dy,$$

$$K_\lambda^*(x) = \frac{1}{4i} \left( \frac{\lambda}{2\pi|x|} \right)^{\frac{n-2}{2}} H_{\frac{n-2}{2}}^{(2)}(\lambda|x|)$$

where  $H_\beta^{(2)}$  is a Hankel function of the second kind and order  $\beta$ . Therefore if  $R_{\partial\Omega}$  is the restriction operator to the boundary of  $\Omega$  we need to prove  $L^2(\Omega) \rightarrow L^2(\partial\Omega)$  estimates for  $R_{\partial\Omega}S_\lambda^+$ . We will do this by constructing an auxiliary quasimode  $v$  defined on  $\mathbb{R}^n$ , for which we know the restriction bounds, then the problem reduces to finding the  $L^2$  norm of  $v$ .

To begin we excise the diagonal of  $(S_\lambda^+)^*$  that is let  $\zeta : \mathbb{R}^n \rightarrow \mathbb{R}^+$  be a smooth cut off function equal to one in  $|x-y| \leq 1$  and supported in  $|x-y| \leq 2$ . Then we decompose  $(S_\lambda^+)^*$  as

$$(2.2) \quad (S_\lambda^+)^* = S^0 + \tilde{S},$$

where

$$(2.3) \quad S^0 u(x) = \int_{\Omega} K_\lambda^*(x-y)\zeta(M^{-1}\lambda(x-y))u(y)dy,$$

$$(2.4) \quad \tilde{S}u(x) = \int_{\Omega} K_\lambda^*(x-y)(1-\zeta(M^{-1}\lambda(x-y)))u(y)dy.$$

We first show that  $S^0$  has better  $L^2(\Omega) \rightarrow L^2(\partial\Omega)$  mapping norm than predicted by Theorem 1.1 and therefore we may focus on the mapping norms  $\tilde{S}$ .

**Proposition 2.1.** *Let  $S^0$  be as defined in (2.3), then*

$$(2.5) \quad \|S^0 u\|_{L^2(\partial\Omega)} \lesssim \lambda^{-\frac{3}{2}} \|u\|_{L^2(\Omega)}.$$

*Proof.* If  $n \geq 3$ , we have that the kernel of  $S^0$ ,  $K^0(x, y)$  has the bounds

$$|K^0(x, y)| \leq |x-y|^{-(n-2)},$$

and is supported in  $|x-y| \leq M\lambda^{-1}$ . Fixing  $x$  we have

$$\begin{aligned} \|K^0(x, \cdot)\|_{L^1} &\lesssim \int_0^{M\lambda^{-1}} r^{-(n-2)} r^{n-1} dr \\ &\lesssim C_M \lambda^{-2}. \end{aligned}$$

Conversely fixing  $y$  we have

$$\|K^0(\cdot, y)\|_{L^1} \lesssim \int_0^{M\lambda^{-1}} r^{-(n-2)} r^{n-2} dr$$

$$\lesssim C_M \lambda^{-1}.$$

Therefore by Young's inequality

$$\|S^0 u\|_{L^2(\partial\Omega)} \lesssim C_M \lambda^{-3/2} \|u\|_{L^2(\Omega)}$$

which is better than (2.5).

In  $\mathbb{R}^2$ , if  $|x - y| \leq M\lambda^{-1}$

$$|K^0(x, y)| \leq \log(\lambda|x - y|) \leq C_\varepsilon (\lambda|x - y|)^{-\varepsilon}$$

for any  $\varepsilon > 0$ . The same application of Young's inequality implies

$$\|S^0 u\|_{L^2(\partial\Omega)} \lesssim C_{M,\varepsilon} \lambda^{-3/2} \|u\|_{L^2(\Omega)}.$$

□

We now focus on the operator  $\tilde{S}$ . Let

$$v = \tilde{S}u,$$

we claim that for  $h = \lambda^{-1}$ ,  $v(x)$  is an  $O_{L^2}(h)$  quasimode for the Laplacian, that is

$$(-h^2 \Delta - 1)v = hf,$$

and is therefore subject to the bounds from [20] and [10],

$$(2.6) \quad \|v\|_{L^2(\partial\Omega)} \lesssim h^{-\frac{1}{4}} [\|v\|_{L^2(\mathbb{R}^n)} + \|f\|_{L^2(\mathbb{R}^n)}]$$

in the general case and

$$(2.7) \quad \|v\|_{L^2(\partial\Omega)} \lesssim h^{-\frac{1}{6}} [\|v\|_{L^2(\mathbb{R}^n)} + \|f\|_{L^2(\mathbb{R}^n)}]$$

in the case where  $\partial\Omega$  is curved. It is known that  $(S_\lambda^+)^*$  has mapping norm  $\lambda^{-1}$  from  $L^2(\Omega) \rightarrow L^2(\mathbb{R}^n)$ , see for example [26]. By the arguments of Proposition 2.1  $S^0$  has mapping norm  $\lambda^{-2}$  from  $L^2(\Omega) \rightarrow L^2(\mathbb{R}^n)$  therefore as

$$\tilde{S} = (S_\lambda^+)^* - S^0,$$

$$\|v\|_{L^2(\mathbb{R}^n)} = \|\tilde{S}u\|_{L^2(\mathbb{R}^n)} \lesssim \lambda^{-1} \|u\|.$$

So to obtain Theorems 1.1 and 1.3 it is enough to show that

$$\|(-h^2 \Delta - 1)v\|_{L^2(\mathbb{R}^n)} \lesssim h^2 \|u\|_{L^2(\Omega)}.$$

Rescaling this to work in terms of  $\lambda$  we require that

$$\|(-\Delta - \lambda^2)v\|_{L^2(\mathbb{R}^n)} \lesssim \|u\|_{L^2(\Omega)}.$$

Now

$$v(x) = \int_{\Omega} K_\lambda^*(x - y) (1 - \zeta(M^{-1}\lambda(x - y))) u(y) dy,$$

where

$$(-\Delta - \lambda^2)K_\lambda^* = \delta.$$

So

$$(2.8) \quad (-\Delta - \lambda^2)v = \int_{\Omega} (1 - \zeta(M^{-1}\lambda(x - y))) [(-\Delta_x - \lambda^2)K_\lambda^*(x - y)] u(y) dy + Eu,$$

$$(2.9) \quad Eu = \int_{\Omega} [-K_\lambda^*(x - y) \Delta_x (1 - \zeta(M^{-1}\lambda(x - y))) - 2\nabla_x (1 - \zeta(M^{-1}\lambda(x - y))) \cdot \nabla_x K_\lambda^*(x - y)] u(y) dy.$$



The first term in (2.8) is zero as the support of  $(1 - \zeta(M^{-1}\lambda(x - y)))$  is bounded away from the diagonal  $x = y$ . The second term is the error term and has kernel supported in  $M\lambda^{-1} \leq |x - y| \leq 2M\lambda^{-1}$ . It therefore suffices to show that

$$\|Eu\|_{L^2(\mathbb{R}^n)} \lesssim \|u\|_{L^2(\Omega)}.$$

**Proposition 2.2.** *If  $E$  is given by (2.9), then*

$$\|Eu\|_{L^2(\mathbb{R}^n)} \lesssim \|u\|_{L^2(\Omega)}.$$

*Proof.* This is similar to the proof of Proposition 2.1. By choosing  $M$  large enough we may assume that the argument of the Hankel function  $\lambda|x - y|$  is large and therefore

$$|H_\beta^{(2)}(\lambda|x - y|)| \leq \lambda^{-\frac{1}{2}}|x - y|^{-\frac{1}{2}}$$

for any  $\beta$ . Therefore on the support of the kernel of  $E$  we have

$$\begin{aligned} |K_\lambda(x - y)| &\lesssim \lambda^{n-2}, \\ |\nabla_x K_\lambda(x - y)| &\lesssim \lambda^{n-1}, \\ |\nabla_x (1 - \zeta(M^{-1}\lambda(x - y)))| &\lesssim \lambda, \\ |\Delta_x (1 - \zeta(M^{-1}\lambda(x - y)))| &\lesssim \lambda^2. \end{aligned}$$

Therefore we have

$$Eu = \int \tilde{K}(x - y)u(y)dy,$$

where  $|\tilde{K}(x - y)| \lesssim \lambda^n$  and is supported on  $M\lambda^{-1} \leq |x - y| \leq 2M\lambda^{-1}$ . Now

$$\|\tilde{K}(\cdot)\|_{L^1} \lesssim \lambda^n \int_0^{2M\lambda^{-1}} r^{n-1}dr \lesssim 1.$$

Therefore by Young's inequality

$$\|Eu\|_{L^2(\mathbb{R}^n)} \lesssim \|u\|_{L^2(\Omega)}$$

as required.  $\square$

### 3. BOUNDEDNESS OF SEMICLASSICAL DOUBLE LAYER POTENTIALS

We now address the mapping norms of the double layer potential operator

$$(3.1) \quad D_\lambda^+ u = \int_{\partial\Omega} \partial_{\nu_y} K_\lambda(x - y)u(y)d\sigma_y.$$

We proceed in a similar fashion as the proof for the single layer potential working instead with the adjoint operator  $(D_\lambda^+)^*$ . Let  $\zeta : \mathbb{R}^n \rightarrow \mathbb{R}$  be a smooth cut off function equal to one in  $|x| \leq 1$  and supported in  $|x| \leq 2$ . Then we decompose  $D_\lambda^*$  as

$$(3.2) \quad (D_\lambda^+)^* = D_0 + \tilde{D}$$

where

$$(3.3) \quad D_0 u = \int_{\Omega} \partial_{\nu_x} K_\lambda^*(x - y) \zeta(M^{-1}\lambda(x - y)) u(y) dy$$

and

$$(3.4) \quad \tilde{D} u = \int_{\Omega} \partial_{\nu_x} K_\lambda^*(x - y) [1 - \zeta(M^{-1}\lambda(x - y))] u(y) dy.$$

Similar to the single layer potential case we will treat  $D_0$  by Young's inequality and  $\tilde{D}$  by quasimode methods.

**Proposition 3.1.** *Let  $D_0$  be as defined in (3.3) then*

$$\|D_0 u\|_{L^2(\partial\Omega)} \lesssim \lambda^{-1/2} \|u\|_{L^2(\Omega)}$$

*Proof.* We have that

$$\begin{aligned} \partial_{\nu_x} K_\lambda^\star(x-y) &= \frac{i}{4} \left( \frac{x-y}{|x-y|} \cdot \nu_x \right) \left[ - \left( \frac{\lambda}{2\pi|x-y|} \right)^{\frac{n-2}{2}} \lambda H_{\frac{n}{2}}^{(2)}(\lambda|x-y|) - \right. \\ &\quad \left. \frac{n-2}{2|x-y|} \left( \frac{\lambda}{2\pi|x-y|} \right)^{\frac{n-2}{2}} H_{\frac{n-2}{2}}^{(2)}(\lambda|x-y|) - \lambda^{-1} \left( \frac{\lambda}{2\pi|x-y|} \right)^{\frac{n}{2}} H_{\frac{n-2}{2}}^{(2)}(\lambda|x-y|) \right] \end{aligned}$$

Therefore on the support of the kernel of  $D_0$  we have that

$$|\partial_{\nu_x} K_\lambda^\star(x-y)| \leq |x-y|^{-(n-1)}$$

We cannot directly apply Young's inequality as  $\|K(\cdot, y)\|_{L^1}$  is not bounded. However if we decompose dyadically we may use Young's inequality on each piece and, since  $\|K(x, \cdot)\|_{L^1}$  is much better than  $O(1)$ , recover something summable. Accordingly we write

$$D_0 = \sum_{j=0}^{\infty} D_0^j$$

where

$$(3.5) \quad \begin{aligned} D_0^j u &= \int_{\Omega_j} \partial_{\nu_y} K_\lambda^\star(x-y) \zeta(M^{-1}\lambda(x-y)) u(y) dy \\ \Omega_j &= \Omega \cap \{y \mid 2^{-j} M \lambda^{-1} \leq |x-y| \leq 2^{-j+1} M \lambda^{-1}\} \end{aligned}$$

Now applying Young's inequality to each  $D_0^j$  we have

$$\begin{aligned} \|D_0^j u\|_{L^2(\partial\Omega)} &\lesssim (2^j \lambda)^{n-1} \cdot (2^{-j} \lambda^{-1})^{\frac{n}{2}} (2^{-j} \lambda^{-1})^{\frac{n-1}{2}} \|u\|_{L^2(\Omega)} \\ &\lesssim \lambda^{-1/2} 2^{-j/2} \|u\|_{L^2(\Omega)} \end{aligned}$$

and therefore

$$\|D_0 u\|_{L^2(\partial\Omega)} \lesssim \lambda^{-1/2} \|u\|_{L^2(\Omega)}$$

as claimed. □

**Proposition 3.2.** *Let  $\tilde{D}$  be given by (3.4) then*

$$\|\tilde{D}u\|_{L^2(\partial\Omega)} \lesssim \|u\|_{L^2(\Omega)}$$

*Proof.* We note that if we define the auxiliary function  $w$  by

$$(3.6) \quad w = \tilde{D}u$$

then

$$(3.7) \quad w = \partial_\nu v + Eu$$

where

$$Eu = \frac{\lambda}{M} \int_{\Omega} K_\lambda^\star(x-y) \partial_{\nu_y} \zeta(M^{-1}\lambda(x-y)) u(y) dy$$

and  $v$  is the quasimode

$$v = \tilde{S}u$$

From Tacy [21] we know that normal derivatives of quasimodes enjoy the hypersurface restriction bound

$$\|\partial_\nu v\|_{L^2(\partial\Omega)} \lesssim \lambda \|v\|_{L^2(\Omega)}$$

therefore by the  $L^2(\Omega) \rightarrow L^2(\mathbb{R}^n)$  mapping properties of the single layer potential operator

$$(3.8) \quad \|\partial_\nu v\|_{L^2(\partial\Omega)} \lesssim \|u\|_{L^2(\Omega)}.$$

So we can restrict our attention to  $Eu$ . We write

$$Eu = \int_{\Omega} E(x, y) u(y) dy$$

and note by Young's inequality

$$\|Eu\|_{L^2(\partial\Omega)} \lesssim \sup_{x, y} \|E(x, \cdot)\|_{L^1}^{1/2} \|E(\cdot, y)\|_{L^1}^{1/2} \|u\|_{L^2(\Omega)}$$

On the support of  $E(x, y)$  we have that

$$\begin{aligned} |K^*(x - y)| &\lesssim \lambda^{n-2} & n \neq 2 \\ |K^*(x - y)| &\lesssim \log \lambda & n = 2 \end{aligned}$$

Therefore for  $n \neq 2$  we have

$$\sup_x \|E(x, \cdot)\| \lesssim \lambda^{n-1} \cdot \lambda^{-n} \lesssim \lambda^{-1}$$

and

$$\sup_y \|E(\cdot, y)\| \lesssim \lambda^{n-1} \cdot \lambda^{-(n-1)} \lesssim 1$$

For  $n = 2$

$$\sup_x \|E(x, \cdot)\| \lesssim \lambda \log \lambda \cdot \lambda^{-2} \lesssim \lambda^{-1} \log \lambda$$

and

$$\sup_y \|E(\cdot, y)\| \lesssim \lambda \log \lambda \cdot \lambda^{-1} \lesssim \log \lambda$$

so for any  $\epsilon > 0$

$$\|Eu\|_{L^2(\partial\Omega)} \lesssim \lambda^{\frac{-1+\epsilon}{2}} \lambda^{\frac{\epsilon}{2}} \|u\|_{L^2(\Omega)}$$

and therefore setting  $\epsilon < 1/2$

$$\|Eu\|_{L^2(\partial\Omega)} \lesssim \|u\|_{L^2(\Omega)}$$

as required. □

#### 4. SHARP EXAMPLES AND FURTHER REMARKS

In this section, we construct examples to show that the estimates in Theorems 1.1, 1.3, and 1.4 are in fact sharp. That is, we prove that the equalities hold in some domains for some sequences of functions. Furthermore, we make some remarks on single layer potentials and spectral projection operators in strictly convex domains.

**4.1. Sharpness of Theorem 1.1: Semiclassical single layer potentials in general domains.** In the view of the Stone's formula (1.6) and Corollary 1.2, we only need to prove the sharpness of  $dE_\lambda$ , and then the sharpness of  $S_\lambda$  follows evidently. In fact, we show that

$$(4.1) \quad \frac{\|dE_\lambda(f_\lambda d\sigma)\|_{L^2(\Omega)}}{\|f_\lambda\|_{L^2(\partial\Omega)}} \geq c\lambda^{-\frac{3}{4}}$$

in a square for some constant  $c$  and functions  $\{f_\lambda\}$ . Throughout this subsection, we denote  $x = (x', x_n) \in \mathbb{R}^n$  where  $x' \in \mathbb{R}^{n-1}$  and  $x_n \in \mathbb{R}$ . First we observe the following fact.

**Lemma 4.1.** *Write  $\Omega' = [-1, 1] \times \cdots \times [-1, 1] \subset \mathbb{R}^{n-1}$ , for  $\lambda \geq 0$  there exists an  $L^2$  normalized function  $f_\lambda$  such that*

- (1)  $\text{supp } f_\lambda \subset \Omega'$ ,
- (2)  $\hat{f}_\lambda \geq 0$ ,
- (3)  $\hat{f}_\lambda(\xi') \geq c_1$  if  $|\xi' - \eta'_\lambda| \leq 1$  for  $\eta'_\lambda = (\lambda, 0, \dots, 0)$  and some positive constant  $c_1$  depending only on the dimension.

*Proof of Lemma 4.1.* Fix a Schwartz function  $\varphi$  such that  $\hat{\varphi} \geq 0$  and  $\hat{\varphi} = 1$  in  $\Omega'$ . Let  $f_0 = \varphi \chi_{\Omega'} / \|\varphi \chi_{\Omega'}\|_{L^2(\mathbb{R}^{n-1})}$ , obviously  $\text{supp } f_0 \subset \Omega'$ , and we verify that  $f_0$  also satisfies (2) and (3) above. Since  $\chi_{\Omega'}$  is even, and

$$\hat{\chi}_{\Omega'}(\xi') = \prod_{i=1}^{n-1} \frac{2 \sin(\xi_i)}{\xi_i},$$

therefore,  $\hat{f}_0 = c \hat{\varphi} * \hat{\chi}_{\Omega'} \geq 0$ . Compute that

$$\hat{f}_0(\xi') = c \int_{\mathbb{R}^{n-1}} \hat{\varphi}(\eta') \hat{\chi}_{\Omega'}(\xi' - \eta') d\eta' \geq c \int_{|\eta'| \leq 1} \hat{\chi}_{\Omega'}(\xi' - \eta') d\eta' \geq c_1,$$

if  $|\xi' - \eta'_0| \leq 1$ . Then one only needs to set  $f_\lambda(x') = e^{ix' \cdot \eta'_\lambda} f_0(x')$ .  $\square$

Denote  $\Omega = \Omega' \times [0, 1] \subset \mathbb{R}^n$ , and let  $\text{supp } f_\lambda \subset \Omega' \times \{x_n = 0\}$ . Furthermore, we need the following lemma.

**Lemma 4.2.** *There exists a function  $g$  such that*

- (1)  $\text{supp } g \subset \Omega$ ,
- (2)  $0 \leq g \leq c_2$  for some constant  $c_2 > 0$  depending only on the dimension,
- (3)  $\hat{g}(\xi) \geq c_3$  in  $\{|\xi| \leq c_4\}$  for some positive constants  $c_3$  and  $c_4 \leq \frac{1}{2}$  depending only on the dimension.

*Proof of Lemma 4.2.* Fix  $z = (0, \dots, 0, \frac{1}{2}) \in \Omega$ , and write  $\varphi = \chi_{|x-z| \leq \frac{1}{4}}$ . Let  $g = \varphi * \varphi$ , then both (1) and (2) above are satisfied, and

$$\hat{g}(0) = [\hat{\varphi}(0)]^2 = \left[ \int_{\mathbb{R}^n} \varphi(x) dx \right]^2 > 0.$$

Thus  $\hat{g}(\xi) \geq c_3$  in  $\{|\xi| \leq c_4\}$  because  $\hat{g}$  is continuous.  $\square$

In order to evaluate  $\|dE_\lambda(f_\lambda d\sigma)\|_{L^2(\Omega)}$ , notice that  $\widehat{f_\lambda d\sigma}(\xi) = \hat{f}_\lambda(\xi')$  and

$$[dE_\lambda(f_\lambda d\sigma)]^\wedge(\xi) = \delta(|\xi|^2 - \lambda^2) \widehat{f_\lambda d\sigma}(\xi) = \frac{2\hat{f}_\lambda(\xi') d\mu}{\lambda},$$

where  $d\mu$  is the surface measure on  $\{|\xi| = \lambda\}$ . Using the function  $g$  constructed in Lemma 4.2, for any  $\xi$  such that

$$\xi \in \left\{ 0 \leq \lambda - |\xi| \leq \frac{c_4}{2} \text{ and } |\xi' - \eta'_\lambda| \leq \frac{1}{2} \right\} := G_\lambda,$$

we have

$$[dE_\lambda(f_\lambda d\sigma)]^\wedge * \hat{g}(\xi) \geq \frac{2c_3}{\lambda} \int_{\{|\eta|=\lambda\} \cap \{|\eta-\xi| \leq c_4\}} \hat{f}_\lambda(\eta') d\mu \geq \frac{c \cdot c_1 \cdot c_3 \cdot c_4^{n-1}}{\lambda},$$

in which we use the geometric fact that the area measure  $|\{|\eta| = \lambda\} \cap \{|\eta - \xi| \leq c_4\}| \sim c_4^{n-1}$  if  $\xi$  is a fixed point near the sphere with  $0 \leq \lambda - |\xi| \leq \frac{c_4}{2} < \frac{1}{4}$ . Recall that  $\eta'_\lambda = (\lambda, 0, \dots, 0)$ , then the volume measure of  $G_\lambda$  has

$$|G_\lambda| = \left| \left\{ 0 \leq \lambda - |\xi| \leq \frac{c_4}{2} \text{ and } |\xi' - \eta'_\lambda| \leq \frac{1}{2} \right\} \right| \sim \sqrt{\lambda}.$$

Therefore,

$$\|dE_\lambda(f_\lambda d\sigma)g\|_{L^2(\mathbb{R}^n)} = c \| [dE_\lambda(f_\lambda d\sigma)]^\wedge * \hat{g} \|_{L^2(\mathbb{R}^n)} \geq c\lambda^{-1} |G_\lambda|^{\frac{1}{2}} \geq c\lambda^{-\frac{3}{4}}.$$

However, since  $g$  is supported in  $\Omega$  and bounded from above,

$$\|dE_\lambda(f_\lambda d\sigma)\|_{L^2(\Omega)} = \|dE_\lambda(f_\lambda d\sigma)\chi_\Omega\|_{L^2(\mathbb{R}^n)} \geq c_2^{-1} \|dE_\lambda(f_\lambda d\sigma)g\|_{L^2(\mathbb{R}^n)} \geq c\lambda^{-\frac{3}{4}},$$

and (4.1) is proved.

#### 4.2. Sharpness of Theorem 1.3: Semiclassical single layer potentials in curved domains.

Similarly as in Section 4.1, we only need to prove the sharpness of  $dE_\lambda$ , and we show that

$$(4.2) \quad \frac{\|dE_\lambda(f_\lambda d\sigma)\|_{L^2(\Omega)}}{\|f_\lambda\|_{L^2(\partial\Omega)}} \geq c\lambda^{-\frac{5}{6}}$$

in an annulus for some constant  $c$  and functions  $\{f_\lambda\}$ .

Let  $B_1 = \{x \in \mathbb{R}^2, |x| < 1\}$ ,  $B_2 = \{x \in \mathbb{R}^2, |x| < 2\}$ , and  $\Omega = \{x \in \mathbb{R}^2, 1 < |x| < 2\}$ . Write

$$f_k(x) = e^{ik\theta} \in L^2(\partial\Omega)$$

when  $r = 1$ . Here,  $x = (r \cos \theta, r \sin \theta)$ . Then

$$u(x) = dE_\lambda(f_k d\sigma)(x) = aJ_k(\lambda r)e^{ik\theta},$$

in which  $J_k$  is the Bessel function of the first kind and order  $k$ . We pick  $\lambda = j_{k,1}$  as the first positive zero of  $J_k$ . Then  $u$  solves the Dirichlet boundary value problem

$$\begin{cases} -\Delta u = \lambda^2 u & \text{in } B_1, \\ u = 0, \partial_r u = e^{ik\theta} & \text{on } \partial B_1. \end{cases}$$

From [5, Section 10.21.40],  $J'_k(\lambda) = O(k^{-\frac{2}{3}})$ . Thus

$$(4.3) \quad a = \frac{1}{\lambda J'_k(\lambda)} = O(k^{-\frac{1}{3}}).$$

We also have

$$(4.4) \quad \lambda = k + c_5 k^{\frac{1}{3}} + O(k^{-\frac{1}{3}}),$$

where  $c_5 = 1.86\dots$  is an independent constant. Since  $(\Delta + \lambda^2)u = 0$ ,

$$\left[ \partial_r^2 + \frac{1}{r} \partial_r + \left( \lambda^2 - \frac{k^2}{r^2} \right) \right] u = 0.$$

Furthermore,  $\partial_r u(x) = a\lambda J'_k(\lambda r)e^{ik\theta}$  and  $\partial_r^2 u(x) = a\lambda^2 J''_k(\lambda r)e^{ik\theta}$ . To evaluate  $\|u\|_{L^2(\Omega)}$ , notice that in  $\mathbb{R}^n$ ,

$$\Delta = \partial_r^2 + \frac{n-1}{r} \partial_r + \frac{1}{r^2} \Delta_{\mathbb{S}^{n-1}},$$

in which  $\Delta_{\mathbb{S}^{n-1}}$  is the Laplacian on the sphere  $\mathbb{S}^{n-1}$ . Then the commutator

$$\begin{aligned}
[\Delta, r\partial_r] &= \left[ \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}}, r\partial_r \right] \\
&= [\partial_r^2, r\partial_r] + \left[ \frac{n-1}{r}\partial_r, r\partial_r \right] + \left[ \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}}, r\partial_r \right] \\
&= 2\partial_r^2 + \frac{2(n-1)}{r}\partial_r + \frac{2}{r^2}\Delta_{\mathbb{S}^{n-1}} \\
&= 2\Delta.
\end{aligned}$$

Using the above facts and Green's formula, we have

$$\begin{aligned}
-2\lambda^2 \int_{|x|<R} |u|^2 &= 2 \int_{|x|<R} \Delta u \cdot \bar{u} = \int_{|x|<R} [\Delta, r\partial_r] u \cdot \bar{u} \\
&= \int_{|x|<R} [\Delta + \lambda^2, r\partial_r] u \cdot \bar{u} \\
&= \int_{|x|<R} (\Delta + \lambda^2)(r\partial_r u) \cdot \bar{u} - r\partial_r u \cdot (\Delta + \lambda^2)\bar{u} \\
&= \int_{|x|=R} \partial_r(r\partial_r u) \cdot \bar{u} - r\partial_r u \cdot \partial_r \bar{u} \\
&= \int_{|x|=R} \partial_r u \cdot \bar{u} + r\partial_r^2 u \cdot \bar{u} - r|\partial_r u|^2 \\
&= \int_{|x|=R} [-r(\lambda^2 - k^2 r^{-2})] u \cdot \bar{u} - r|\partial_r u|^2 \\
&= -2a^2\pi R^2 [(\lambda^2 - k^2 R^{-2})(J_k(\lambda R))^2 + \lambda^2 (J'_k(\lambda R))^2],
\end{aligned}$$

which implies

$$(4.5) \quad \int_{|x|<R} |u|^2 = a^2\pi R^2 \left[ \left(1 - \frac{k^2}{\lambda^2 R^2}\right) (J_k(\lambda R))^2 + (J'_k(\lambda R))^2 \right].$$

If  $R = 1$ , then note that  $\lambda$  is a zero of  $J_k$ , by  $|a| \sim \lambda^{-\frac{1}{3}}$  in (4.3) and  $J'_k(\lambda) = J'_k(j_{k,1}) \sim \lambda^{-\frac{2}{3}}$  in [5, Section 10.21.40],

$$(4.6) \quad \|u\|_{L^2(B_1)} = \left( \int_{|x|<1} |u|^2 \right)^{\frac{1}{2}} = |a|\sqrt{\pi}|J'_k(\lambda)| = c\lambda^{-1}.$$

If  $R = 2$ , then

$$(4.7) \quad \|u\|_{L^2(B_2)} = 2|a|\sqrt{\pi} \left[ \left(1 - \frac{k^2}{4\lambda^2}\right) (J_k(2\lambda))^2 + (J'_k(2\lambda))^2 \right]^{\frac{1}{2}} \geq c\lambda^{-\frac{5}{6}}.$$

Here, we use the asymptotic expansions of Bessel functions for large orders in [5]

$$J_k(k \sec \beta) \sim \left( \frac{1}{k \tan \beta} \right)^{\frac{1}{2}} \cos \left( k \tan \beta - k\beta - \frac{1}{4}\pi \right),$$

and

$$J'_k(k \sec \beta) \sim \left( \frac{\sin(2\beta)}{k} \right)^{\frac{1}{2}} \sin \left( k \tan \beta - k\beta - \frac{1}{4}\pi \right),$$

in which  $\sec \beta = 2\lambda/k \rightarrow 2$  in the view of (4.4), and thus  $\beta \sim \frac{\pi}{3}$ . Therefore,

$$\|u\|_{L^2(\partial\Omega)} = \|u\|_{L^2(B_2)} - \|u\|_{L^2(B_1)} \geq c\lambda^{-\frac{5}{6}},$$

and the example is completed.

**4.3. Further remark: Semiclassical single layer potentials in strictly convex domains.**

If we choose  $R = 1 + \varepsilon$  in (4.5) for any fixed  $\varepsilon > 0$ , then

$$(4.7') \quad \|u\|_{L^2(B_{1+\varepsilon})} \geq c\lambda^{-\frac{5}{6}}$$

is valid when  $k$ , and therefore  $\lambda$ , is large. (We can argue similarly by setting  $\beta$  asymptotically fixed depending only on  $\varepsilon$ .) Comparing (4.6) and (4.7)', we see that the  $L^2$  norm of  $u$  is essentially concentrated outside the disc. As we explained in the Introduction, this is because the semiclassical singularities of  $u$  which propagate along the tangent lines of the circle dominate the estimates, and they all lie outside of  $B_1$ , when restricted in  $B_1$ , the  $L^2$  norm of  $u$  is smaller ( $\sim \lambda^{-1}$ ), and when restricted in  $B_{1+\varepsilon} \setminus B_1$ , the  $L^2$  norm of  $u$  is larger ( $\sim \lambda^{-\frac{5}{6}}$ ).

However, in the case when the domain is flat as in Section 4.1, the tangent lines coincide with the boundary, one may get worse  $L^2$  bound of  $u$  ( $\sim \lambda^{-\frac{3}{4}}$ ).

The above observation motivates us to consider the problem in strictly convex domains, in which case all the tangent lines lie outside the domain. Analogously to the unit disc, we make the following conjecture.

**Conjecture 4.3.** *If  $\Omega$  is strictly convex, then*

$$(4.8) \quad \|S_\lambda(f)\|_{L^2(\Omega)} \leq c\lambda^{-1}\|f\|_{L^2(\partial\Omega)},$$

and

$$(4.9) \quad \|dE_\lambda(f d\sigma)\|_{L^2(\Omega)} \leq c\lambda^{-1}\|f\|_{L^2(\partial\Omega)}.$$

The computation in Section 4.2 already gave the sharp example in this case once one observes (4.6), which says

$$\frac{\|dE_\lambda(f_\lambda d\sigma)\|_{L^2(\Omega)}}{\|f_\lambda\|_{L^2(\partial\Omega)}} = c\lambda^{-1}$$

is valid in the unit ball for some constant  $c$  and functions  $\{f_\lambda\}$ .

In fact, the estimates in Conjecture 4.3 are sharp in any strictly convex domain: From [1, 11, 12],

$$\|u\|_{L^2(\Omega)} \approx \lambda^{-1}\|\partial_\nu u\|_{L^2(\partial\Omega)},$$

where  $u$  is a Dirichlet eigenfunction in  $\Omega$ , therefore  $u = S_\lambda(\partial_\nu u)$ , and (4.8) and (4.9) are sharp in all strictly convex domains.

**4.4. Sharpness of Theorem 1.4: Semiclassical double layer potentials.** We show the sharpness in the unit ball:

$$(4.10) \quad \frac{\|D_\lambda(f_\lambda d\sigma)\|_{L^2(\Omega)}}{\|f_\lambda\|_{L^2(\partial\Omega)}} \geq c$$

for some constant  $c$  and functions  $\{f_\lambda\}$ .

Consider the Neumann eigenfunctions:

$$\begin{cases} -\Delta u = \lambda^2 u & \text{in } B_1, \\ \partial_\nu u = 0, \quad u = e^{ik\theta} & \text{on } \partial B_1, \end{cases}$$

then adopting the same notations as in Section 4.2, we have

$$u(x) = D_\lambda(f_k d\sigma)(x) = aJ_k(\lambda r)e^{ik\theta},$$

in which  $\lambda = j'_{k,l}$  is the  $l$ -th zero of  $J'_k$ , and  $a = 1/J_k(\lambda)$ .

Let (4.5) apply to  $R = 1$  and  $J'_k(\lambda) = 0$ :

$$\|u\|_{L^2(B_1)} = \left( \int_{|x|<1} |u|^2 \right)^{\frac{1}{2}} = \sqrt{\pi \left( 1 - \frac{k^2}{\lambda^2} \right)} \geq c,$$

by picking  $\lambda \sim 2k$ . In fact,  $\lambda = j'_{k,l} \rightarrow \infty$  as  $l \gg k \rightarrow \infty$ .

On the other hand, to achieve the equality in (1.7):

$$\|u\|_{L^2(\partial B_1)} \lesssim \lambda^{\frac{1}{3}} \|u\|_{L^2(B_1)},$$

or

$$\|u\|_{L^2(B_1)} \gtrsim \lambda^{-\frac{1}{3}}$$

because  $\|u\|_{L^2(\partial B_1)} = \|f\|_{L^2(\partial B_1)} = O(1)$ . We pick  $\lambda = j'_{k,1}$  as the first zero of  $J'_k$ . From [5, Section 10.21.40], we have

$$\lambda = j'_{k,1} = k + O(k^{\frac{1}{3}}),$$

then

$$\|u\|_{L^2(B_1)} = \sqrt{\pi \left( 1 - \frac{k^2}{(j'_{k,1})^2} \right)} = O(k^{-\frac{1}{3}}) = O(\lambda^{-\frac{1}{3}}),$$

achieving the quality in (1.7). See also [11, Example 7] on the boundary estimates of Neumann eigenfunctions.

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